Journal of Statistical Physics, Vol. 58, Nos. 5/6, 1990

Are There More Than Five Linearly-Independent Collision Invariants for the Boltzmann Equation?

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Received July 6, 1989; revision received September 28, 1989

The problem of finding the summational collision invariants for the Boltzmann equation is tackled with the aim of proving that the most general solution of the problem is not different from the standard one even when the equation defining a collision invariant ψ is only satisfied almost everywhere in $R^3 \times R^3 \times S^2$. The collision invariant ψ is assumed to be in the Hilbert space H_ω of the functions which are square integrable with respect to a Maxwellian weight.

KEY WORDS: Kinetic theory; Boltzmann equation; collision invariant.

1. INTRODUCTION

One of the basic ingredients of the kinetic theory of a monatomic rarefied gas[1,2] is the concept of a (summational) collision invariant, i.e., a function $\psi(\xi)$ such that

$$\psi(\xi'_*) + \psi(\xi') - \psi(\xi_*) - \psi(\xi) = 0 \tag{1.1}$$

where ξ, ξ_* are vectors in R^3 (with the physical meaning of molecular velocities) and

$$\begin{aligned} \xi' &= \xi - \mathbf{n}(\mathbf{n} \cdot \mathbf{V}) \\ \xi'_* &= \xi'_* + \mathbf{n}(\mathbf{n} \cdot \mathbf{V}) \end{aligned} \tag{1.2}$$

Here $\mathbf{V} = \xi - \xi_*$ is the relative velocity and $\mathbf{n}$ is a unit vector. Equation (1.1) must be satisfied almost everywhere in $R^3 \times R^3 \times S^2$.

Equation (1.1) plays an important role in several problems of kinetic

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0022-4715/90/0300-0817$06.00/0

theory; in particular, if f is the distribution function and $\psi = \log f$, then Eq. (1.1) must be satisfied by all the possible equilibrium solutions.

The first discussion of Eq. (1.1) was due to Boltzmann,[3,4] who assumed ψ to be differentiable twice and arrived at the result that the most general solution of Eq. (1.1) is given by

$$\psi(\xi) = A + \mathbf{B} \cdot \xi + C\,|\xi|^2 \tag{1.3}$$

where $A \in R$, $\mathbf{B} \in R^3$, $C \in R$ are arbitrary constants. This result seems to be physically obvious because the solutions of Eq. (1.1) form a linear manifold and if more than five collision invariants existed, it would appear that ξ' and ξ'_* (the velocities which give rise to ξ and ξ_* through an elastic impact) would depend on less parameters than the two scalar ones specified by $\mathbf{n}$. This would, of course, be at variance with the fact that the centers of the colliding particles at the moment of the impact lie on a line which can have any direction in space. Yet, there are good reasons for trying to give a detailed proof of the fact that there are no other solutions in addition to those specified by Eq. (1.3); in fact, the hypothetical new solutions might be linearly independent but functionally dependent on the previous ones. Indeed, it is remarkable that Boltzmann was not satisfied with physical evidence and felt the necessity of giving the above-mentioned proof.

After Boltzmann, the matter of finding the solutions of Eq. (1.1) was investigated by Gronwall[5,6] (who was the first to reduce the problem to Cauchy's functional equation for linear functions), Carleman,[7] and Grad.[8] All these authors assumed ψ to the continuous and proved that it must be of the form given in Eq. (1.3). Slightly different versions of Carleman's proof are given in refs. 2 and 9. In the latter monograph[9] the authors prove that the solution is of the form (1.3), even if the function ψ is assumed to be measurable rather than continuous. In fact, they use a result on the solutions of Cauchy's equation:

$$f(\mathbf{x} + \mathbf{v}) = f(\mathbf{x}) + f(\mathbf{v}) \qquad (\mathbf{x}, \mathbf{v} \in R^n) \tag{1.4}$$

valid for measurable functions. It seems, however, that when passing from continuous to (possibly) discontinuous functions, one should insist on the fact that Eq. (1.1) is satisfied almost everywhere and not everywhere in $R^3 \times R^3 \times S^2$, as assumed in ref. 9. The proof under consideration does not even mention the role played by zero-measure sets in $R^3 \times R^3 \times S^2$. This makes the proof rather uninteresting from the viewpoint taken here. It might be possible, although this will not be attempted in this paper, to transform the proof in ref. 9 into a proof that the collision invariants are the classical ones under the assumption that Eq. (1.1) holds almost everywhere. This would require arguments from the theory of measurable functions which were not even mentioned in ref. 9.

In this paper the problem of solving Eq. (1.1) is tackled with the aim of proving that Eq. (1.3) gives the most general solution of Eq. (1.1), when the latter is satisfied almost everywhere in $R^3 \times R^3 \times S^2$. I shall assume that ψ is in the Hilbert space H_ω of the functions which are square integrable with respect to a Maxwellian weight $\omega(|\xi|) = (\beta/\pi)^{3/2} \exp(-\beta |\xi|^2)$, $\beta > 0$.

After this paper was accepted for publication, it was brought to my attention that a completely different proof of the same result (under the assumption that $\psi \in L^1_{\text{loc}}$) is contained in a paper by Arkeryd.[10]

2. THE COLLISION INVARIANTS MUST BE POLYNOMIALS

Let $\psi \in H_\omega$ and define

$$K\psi = \frac{1}{4\pi} \int_{R^3 \times S^2} \omega(|\xi|_*)[\psi(\xi'_*) + \psi(\xi') - \psi(\xi_*)] \, d\mathbf{n} \, d\xi_* \tag{2.1}$$

where $d\mathbf{n}$ is the measure induced by the Lebesgue measure on the unit sphere S^2. Some properties of K are as follows.

Lemma 1. K is a bounded self-adjoint operator in H_ω.

Proof. Note that, if $(\cdot, \cdot)$ is the scalar product in H_ω, then

$$(\phi, K\psi) = (\phi, \psi) - \frac{1}{16\pi} \int_{R^3 \times S^2} \omega(|\xi|) \, \omega(|\xi_*|)(\psi'_* + \psi' - \psi_* - \psi) \times (\phi'_* + \phi' - \phi_* - \phi) \, d\mathbf{n} \, d\xi_* \, d\xi \tag{2.2}$$

where I simply write ψ'_*, ψ', and ψ_* for $\psi(\xi'_*)$, $\psi(\xi')$, and $\psi(\xi_*)$ and use the fact that $\omega(|\xi|) \, \omega(|\xi_*|) = \omega(|\xi'|) \, \omega(|\xi'_*|)$. Equation (2.2) shows that $(\phi, K\psi) = (K\phi, \psi)$. In addition,

$$-3 \, \|\psi\|^2 \leqslant (\psi, K\psi) \leqslant \|\psi\|^2 \tag{2.3}$$

where the lower bound (which is by no means the best possible one) follows from Eq. (2.1) directly and the upper bound from Eq. (2.2). ∎

Lemma 2. K transforms polynomials of the mth degree into polynomials of degree not larger than m.

Proof. If ψ is a polynomial of the mth degree in ξ, $\psi'_* + \psi' - \psi_*$ is also a polynomial of the mth degree in ξ, with coefficients depending on $\mathbf{n}$. After integration of these coefficients, the result is a polynomial in ξ of degree certainly not higher than m. ∎

Lemma 3. ψ is a collision invariant if and only if ψ is an eigenfunction of K corresponding to the unit eigenvalue.

Proof. The "if" part follows from Eq. (2.2) by noting that if $\phi = \psi$ and $K\psi = \psi$ (a.e. in ξ), then the last integral in Eq. (2.2) must vanish, which is possible only if ψ a.e. satisfies Eq. (1.1). The "only if" part is trivial. ∎

I now prove the following basic result on the operator K:

Theorem 1. There is a complete set of eigenfunctions of K whose elements are polynomials in ξ.

Proof. Let ψ be an eigenfunction of K corresponding to an eigenvalue λ. Expand ψ into a series of polynomials in H_ω (Hermite tensor polynomials or Sonine–Laguerre polynomials):

$$\psi = \sum_m a_m P_m \tag{2.4}$$

where m is a triplet (m_1, m_2, m_3) of nonnegative integers which identify the polynomials. The degree of P_m is denoted by $|m|$ and $(P_m, P_l) = \delta_{ml}$. Then

$$\sum_m a_m K P_m = \lambda \sum_m a_m P_m \tag{2.5}$$

and for any triplet j

$$\sum_m a_m (P_j, KP_m) = \lambda \sum_m a_m (P_j, P_m) = \lambda a_j \tag{2.6}$$

Now, since, by Lemma 2, KP_m is a polynomial of degree $|m|$ at most, and hence can be expressed as a linear combination of $P_{m'}$, with $|m'| \leqslant |m|$, (P_j, KP_m) vanishes if $|j| > |m|$. Since, however, $(P_j, KP_m) = (KP_j, P_m)$, it vanishes if $|j| < |m|$ as well. It follows that the only terms surviving in Eq. (2.6) are those with $|j| = |m|$. This means that for each degree we have to solve a finite linear system [having a symmetric matrix with elements (P_l, KP_m)]. The order of this system is equal to the maximum number of mutually orthogonal polynomials of a given degree $|l| = |m|$ and, accordingly, there are as many eigenvectors. We can then construct finite sums $\sum_m a_m P_m$ with $|m|$ fixed in order to get the eigenfunctions of K. The functions obtained in this way are a complete set (because for each degree their number equals the maximum number of mutually orthogonal polynomials of that degree). ∎

I am now in the position of proving the basic result of this paper as a corollary.

Corollary. There is a polynomial basis for the subspace of the collision invariants.

This follows from Lemma 3 and the previous theorem. I remark that even if the eigenvalue $\lambda = 1$ of the operator K were infinitely times degenerate (which is not the case, because K turns out to be compact in H_ω), there would still be a polynomial basis for the collision invariants. ∎

3. THE COLLISION INVARIANTS FORM A FIVE-DIMENSIONAL SUBSPACE

It is clear that one can now proceed to using any of the classical proofs which assume ψ to be continuous or even differentiable, in order to show that Eq. (1.3) gives the most general collision invariant.

The following proof (which is similar to Boltzmann's argument[3,4]) is based on an immediate consequence of the results of the previous section, i.e., the fact that we can look for polynomial solutions (1.1) and hence, *a fortiori*, for a C^2 solution. I start from the remark[2,5,7,9] that since the transformation (1.2) is the most general one leaving $\xi + \xi_*$ and $|\xi|^2 + |\xi_*|^2$ invariant, one must have

$$\psi(\xi) + \psi(\xi_*) = f(\mathbf{x}, u) \tag{3.1}$$

where

$$\mathbf{x} = \xi + \xi_*; \qquad u = \tfrac{1}{2}(|\xi|^2 + |\xi_*|^2) \tag{3.2}$$

If we differentiate Eq. (3.1) with respect to ξ and subtract from the result the analogous derivative with respect to ξ_*, we obtain

$$\frac{\partial\psi}{\partial\xi} - \frac{\partial\psi}{\partial\xi_*} = \frac{\partial f}{\partial u}(\xi - \xi_*) \tag{3.3}$$

where the $*$ in $\psi(\xi_*)$ is omitted when it is differentiated with respect to its own argument. Equation (3.3) implies

$$\left(\frac{\partial\psi}{\partial\xi_i} - \frac{\partial\psi}{\partial\xi_{*i}}\right)(\xi_k - \xi_{*k}) = \left(\frac{\partial\psi}{\partial\xi_k} - \frac{\partial\psi}{\partial\xi_{*k}}\right)(\xi_i - \xi_{*i}) \qquad (i, k = 1, 2, 3) \tag{3.4}$$

If we now differentiate with respect to ξ_j, we obtain

$$\left(\frac{\partial\psi}{\partial\xi_i} - \frac{\partial\psi}{\partial\xi_{*i}}\right)\delta_{kr} + \frac{\partial^2\psi}{\partial\xi_i\,\partial\xi_r}(\xi_k - \xi_{*k}) = \left(\frac{\partial\psi}{\partial\xi_k} - \frac{\partial\psi}{\partial\xi_*{}^k}\right)\delta_{ir} + \frac{\partial^2\psi}{\partial\xi_k\,\partial\xi_r}(\xi_i - \xi_{*i}) \tag{3.4}$$

where δ_{kr} denotes the Kronecker delta. A further differentiation with respect to ξ_{*j} gives

$$\frac{\partial^2\psi}{\partial\xi_i\,\partial\xi_r}\delta_{kj}+\frac{\partial^2\psi}{\partial\xi_{*i}\,\partial\xi_{*j}}\delta_{kr}=\frac{\partial^2\psi}{\partial\xi_k\,\partial\xi_r}\delta_{ij}+\frac{\partial^2\psi}{\partial\xi_{*k}\,\partial\xi_{*j}}\delta_{ir} \tag{3.5}$$

If we let i, k, r take three different values (say $i=1$, $k=2$, $r=3$) and $j=k$, we obtain

$$\frac{\partial^2\psi}{\partial\xi_i\,\partial\xi_r}=0 \qquad (i, r=1, 2, 3;\, i\neq r) \tag{3.6}$$

If we now take $i=r$, $k=j$, $i\neq k$, we obtain

$$\frac{\partial^2\psi}{\partial\xi_i^2}=\frac{\partial^2\psi}{\partial\xi_{*k}^2} \qquad (i\neq k) \tag{3.7}$$

Since the right-hand side cannot depend on $\boldsymbol{\xi}$, we conclude that both sides are constant; this constant does not depend on the index, because we can change the values of i and k, while keeping $i\neq k$. From Eqs. (3.6) and (3.7) we thus conclude that

$$\frac{\partial^2\psi}{\partial\xi_i\,\partial\xi_r}=2A\delta_{ir} \qquad (i, r=1, 2, 3;\, A=\text{const}) \tag{3.8}$$

Equation (3.8) immediately delivers Eq. (1.3).

ACKNOWLEDGMENT

The draft of the present paper was written during a short visit of the author at the Courant Institute for Mathematical Sciences of New York University. The hospitality of the Institute and a useful discussion with Russel Caflisch on the subject matter of this paper are gratefully acknowledged.

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